Root vectors

Vanni Noferini[1]

[1] *Aalto University, Finland*
E-mail: `vanni.noferini@aalto.fi`

Abstract

Root vectors are a classical, albeit somewhat underappreciated, topic in linear algebra in the regular case [4, 11]. More recently, they have proven to be a powerful tool in the singular case as well [1, 2, 8, 9, 10]. In this presentation, we will explore their applicability in increasingly general situations, including classical eigenvalue problems, generalized eigenvalue problems [9], polynomial and rational matrices [1, 2, 4, 10], as well as analytic and meromorphic matrices [8, 11].

Let us first consider the most basic case of classical eigenvalue problems, focusing on root vectors that are polynomials. If $\mathbb{F}$ is an algebraically closed field and $A \in \mathbb{F}^{n\times n}$ has an eigenvalue $\lambda \in \mathbb{F}$, a *root polynomial* for A at λ of order ℓ is defined as a vector $v(x) \in \mathbb{F}[x]^n$ such that

$$(1)\ (A - xI)v(x) = (x - \lambda)^\ell w(x) \text{ with } w(\lambda) \neq 0; \qquad (2)\ v(\lambda) \neq 0.$$

Such a root polynomial can be seen as a generating function for a Jordan chain of A at λ. Indeed, if we expand it as $v(x) = v_0 + v_1(x - \lambda) + v_2(x - \lambda)^2 + v_3(x - \lambda)^3 + \dots$, then it is easy to see that $v_0, v_1, \dots, v_{\ell-1}$ is a Jordan chain of length ℓ for A associated with the eigenvalue λ. A kind of converse statement also holds: for example, if v_0, v_1, v_2 is a Jordan chain of lenght 3 at λ then one has $(A - \lambda I)v_0 = 0$, $(A - \lambda I)v_1 = v_0$, and $(A - \lambda I)v_2 = v_1$. Hence, $(A - xI)(v_0 + (x - \lambda)v_1 + (x - \lambda)^2 v_2) = (x - \lambda)^3(-v_2)$ so that $v(x) = v_0 + (x - \lambda)v_1 + (x - \lambda)^2 v_2$ is a root polynomial of order 3 for A at λ.

One can further extend this idea by constructing *maximal sets* of root polynomials, which correspond to generating functions for canonical sets of Jordan chains. This process involves several steps:

1. A set of root polynomials $\{v_i(x)\}_{i=1}^s$ at λ for A, of orders $\ell_1 \geq \dots \geq \ell_s$, is called λ-independent if the constant matrix $\begin{bmatrix} v_1(\lambda) & \dots & v_s(\lambda) \end{bmatrix}$ has full column rank;

2. A λ-independent set of root polynomials at λ for A is called complete if there are not λ-independent sets of larger cardinality;

3. A complete set of root polynomials at λ for A is called maximal if it cannot be modified by replacing one root polynomial with another of larger order while still maintaining the completeness property.

It can be proven that the orders of a maximal set of root polynomials are precisely the partial multiplicities of the eigenvalue λ for the matrix A. Thus, a maximal set serves as a condensed source of relevant information about the eigenvalue λ, including partial multiplicities and (generalized) right eigenvectors.

The concept of a maximal set of root vectors, which include root polynomials as a special case, can be extended beyond the classical eigenvalue problem represented by the pencil $A - xI$. Specifically, maximal sets of root vectors exist, and exhibit similar properties to those discussed earlier in the context of the generalized eigenvalue problem $(A + xB)$, the polynomial eigenvalue problem $(P(x) \in \mathbb{F}[x]^{m\times n})$, the rational eigenvalue problem $(R(x) \in \mathbb{F}(x)^{m\times n})$, and other nonlinear eigenvalue problems involving matrices over the ring of analytic functions or the field of meromorphic functions.

All these generalizations, unlike the classical eigenvalue problem, encompass the *singular* case. For instance, a pencil $A + xB$ is regular if it is square and $\det(A + xB) \not\equiv 0$, while it is singular otherwise. Analogous definitions apply to polynomial, rational, analytic, and meromorphic matrices. The application of the concept of a canonical set of Jordan chains becomes problematic in the singular case, as it is not immediately clear how to extend the definition of eigenvectors (let alone chains). However, the notion of a maximal set of root vectors is flexible enough to adapt to singular (linear or nonlinear) eigenvalue problems. The starting point is the generalization of the notion of a root polynomial. Suppose $M(x)$ is a minimal basis [3] for the singular pencil $A + xB \in \mathbb{F}[x]1^{m\times n}$, and define $\ker_\lambda(A + xB)$ as the linear span of the columns of $M(\lambda)$. Then, a vector $v(x) \in \mathbb{F}[x]$ is termed a *root polynomial* for $A + xB$ at λ of order ℓ if:

(1) $(A + xB)v(x) = (x - \lambda)^\ell w(x)$ with $w(\lambda) \neq 0$; (2) $v(\lambda) \notin \ker_\lambda(A + xB)$.

Maximal sets are then defined similarly to the regular case, with the exception that for λ-independence one requires $\begin{bmatrix} M(\lambda) & v_1(\lambda) & \dots & v_s(\lambda) \end{bmatrix}$ to have full rank. It is important to note that, in the regular case (or more generally when the pencil $A + xB$ has full column rank), the block $M(\lambda)$ is empty. One significant application is the rigorous definition of eigenvectors also for singular pencils: an eigenvector is a nonzero element of the quotient space $\ker(A + \lambda B)/\ker_\lambda(A + xB)$. It should be noted that, if $A + xB$ has full column rank, then $\ker_\lambda(A + xB) = \{0\}$ is trivial and the conventional definition of an eigenvector is thus regained. Eigenvectors of singular pencils have numerous computational applications [5, 6, 7].

Finally, root vectors can also be defined for rational matrices, using a connection with valuation theory [10], and for analytic (and meromorphic) matrices, using a connection with module theory [8]. In all these settings, by utilizing root vectors, eigenvectors can still be defined also for matrices that do not have full column rank.

References

[1] F. Dopico, V. Noferini. The $\mathbb{DL}(P)$ vector space of pencils for singular matrix polynomials. Preprint.

[2] F. Dopico, V. Noferini. Root polynomials and their role in the theory of matrix polynomials *Linear Algebra Appl.* 584:37–78, (2020).

[3] G. D. Forney Jr.. Minimal bases of rational vector spaces, with applications to multivariable linear system. *SIAM J. Control* 13, 493–520, (1975).

[4] I. Gohberg, P. Lancaster, L. Rodman. *Matrix Polynomials*, SIAM, 2009. Unabridged republication of book first published by Academic Press in 1982.

[5] M. E. Hochstenbach, C. Mehl and B. Plestenjak. Solving singular generalized eigenvalue problems. Part II: Projection and agumentation. Preprint.

[6] D. Kressner and I. Šain Giblić. Singular quadratic eigenvalue problems: linearization and weak condition numbers. *BIT Numer. Math.* 63, 18 (2023).

[7] M. Lotz and V. Noferini. Wilkinson's bus: Weak condition numbers, with an application to singular polynomial eigenproblems. *Found. Comput. Math* 20, 1439–1473 (2020).

[8] V. Noferini. Invertible bases and root vectors for analytic matrix-valued functions Preprint.

[9] V: Noferini, P. Van Dooren. On computing root polynomials and minimal bases of matrix pencils *Linear Algebra Appl.* 658:86–115, (2023).

[10] V. Noferini, P. Van Dooren. Root vectors of polynomial and rational matrices: theory and computation *Linear Algebra Appl.* 656:510–540, (2023).

[11] V. Trofimov. The root subspace of operators that depend analytically on a parameter. Mat. Issled 3, 117–125, (1968).