The role of the field in some questions of matrix algebra

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Abstract

Given a field $\mathbb{F}$ and a matrix property $\mathcal{P}$, one can investigate the maximum possible dimension of a subspace of $M_{m\times n}(\mathbb{F})$ in which every (non-zero) element has property $\mathcal{P}$, and try to identify those subspaces that attain this maximum. This formulation provides a rich source of interesting problems, many of which have a long and influential history. Sometimes the answers to these questions are independent of the field $\mathbb{F}$, for example if $\mathcal{P}$ is an upper bound on rank, or if $m = n$ and $\mathcal{P}$ is nilpotence. Sometimes the answers are highly dependent on $\mathbb{F}$, for example if $\mathcal{P}$ is a lower bound on rank, or if $m = n$ and $\mathcal{P}$ is non-nilpotence. We will discuss the role of field properties in some of these cases.

Fields that are algebraically closed are great for linear algebra, for example because every square matrix is similar to a unique Jordan canonial form. Fields that are real are also very nice, for example because they admit a distinction between positive and negative elements. Finite fields allow opportunities for counting. Fields that possess extensions of finite degree are excellent too, because such an extension is a finite dimensional vector space, with the extra algebraic machinery of a field multiplcation that plays well with the vector space structure. If $\mathbb{K}$ is a field extension of $\mathbb{F}$ of degree n, then $\mathbb{K}$ is isomorphic as a vector space to any other n-dimensional space over $\mathbb{F}$. It follows that any $\mathbb{F}$-vector space V of dimension n can be endowed with an $\mathbb{F}$-bilinear field multiplication arising from $\mathbb{K}$.

In the talk, we will consider how this idea can be used to uncover large subspaces of various matrix spaces in which rank behaves in a controlled way, specifically over fields which admit cyclic Galois extensions of all degrees. These fields comprise a broad class, including for example all finite fields and all finite extensions of $\mathbb{Q}$.